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On the electrodynamics of the Liénard–Wiechert superpotentials

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Abstract. In classical electrodynamics, the Liénard–Wiechert (LW) potentials reveal their own peculiar properties. That is to say, the LW potentials possess a kind of wave and particle duality: they are the solutions to inhomogeneous wave equations which describe the electromagnetic fields produced by a moving charged particle.

From this point of view, this paper considers the particle-dynamics picture of the LW potentials. The consideration is performed based on the new representation of the LW potentials, which was introduced by Kawaguchi and Murata (1989).

From this consideration, some formulae on the LW potentials, which are similar to those of particle dynamics, are presented.

1. Introduction

Since the Liénard–Wiechert (LW) potentials were first introduced, these potentials have been widely used in electrodynamics. A typical example, in which LW potentials are applied, is Schwinger's formula for a power-spectrum distribution of synchrotron radiation [1]. On the other hand, these potentials have been related to a serious problem in electrodynamics for a long time. The LW potentials predict so-called 'radiation damping' in particle dynamics. It is well known that self-consistent particle dynamics, containing radiation damping, has not yet been discovered.

LW potentials reveal their own peculiar properties in classical electrodynamics. That is to say, LW potentials possess a kind of wave and particle duality: they are the solutions to inhomogeneous wave equations which describe the electromagnetic fields produced by a moving charged particle.

From this point of view, this paper considers the particle-dynamics picture of the LW potentials. The consideration is performed based on the new representation of LW potentials, which is presented by Kawaguchi and Murata [2].

2. New representation of the Liénard–Wiechert potentials and superpotentials

In this section, the new representation of LW potentials and superpotentials are summarized for later reference. Let $A^\mu = (\phi/c, \boldsymbol{A})$ be the LW potentials in the four-dimensional form:

$$A^\mu(ct, x) = \frac{e}{4\pi\varepsilon_0 c^2} \frac{u^\mu(t_r)}{R_\nu(t_r)u^\nu(t_r)}. \tag{1}$$

Here, $y^\mu(t_r) = (ct_r, \boldsymbol{y}(t_r))$ are the four-dimensional coordinates of the moving particle, which carries charge, e, $u^\mu = \gamma(1, d\boldsymbol{y}/c\,dt_r)$ is the four-velocity vector of the particle, γ is defined $(1 - |d\boldsymbol{y}/dt_r|^2/c^2)^{-1/2}$, $R^\mu = (x^\mu - y^\mu(t_r))$ is the displacement vector from the source point y^μ to the observation point $x^\mu = (ct, \boldsymbol{x})$, ε_0 is the dielectric constant and c is the velocity of light. The retarded time t_r, is implicitly defined by the following recursive equation,

$$t_r = t - \frac{|\boldsymbol{x} - \boldsymbol{y}(t_r)|}{c}. \tag{2}$$

Then, there are some formulae relevant to the retarded time [3]

$$\frac{c\partial t_r}{\partial x^\mu} = \frac{\gamma R_\mu}{R_\nu u^\nu} \tag{3}$$

or more generally

$$\begin{aligned} \frac{\partial y^\lambda(t_r)}{\partial x^\mu} &= \frac{c\partial t_r}{\partial x^\mu}\frac{dy^\lambda}{cdt_r} \\ &= \frac{R_\mu u^\lambda}{R_\nu u^\nu}. \end{aligned} \tag{4}$$

It is found from equation (4) that y^μ and A^μ satisfy the following equations:

$$\frac{\partial y^\nu}{\partial x^\nu} = 1 \tag{5}$$

$$A^\mu = \frac{\partial y^\mu}{\partial x^\nu} A^\nu. \tag{6}$$

Moreover,

$$\Box y^\mu = -\frac{2u^\mu}{R_\nu u^\nu} \tag{7}$$

where $\Box(= -\partial^2/\partial x_\nu \partial x^\nu)$ is the D'Alembertian.

Now, comparing equation (1) with equation (7), one can derive the new representation of the LW potentials as follows

$$A^\mu(ct, \boldsymbol{x}) = -\frac{e}{8\pi\varepsilon_0 c}\Box y^\mu. \tag{8}$$

Accordingy, the functions $y^\mu(ct, \boldsymbol{x})$ can be regarded as the potentials of the LW potentials (the LW superpotentials). Furthermore, if we introduce the following tensor $\Pi_{\mu\nu}$:

$$\Pi_{\mu\nu}(ct, \boldsymbol{x}) = \frac{e}{8\pi\varepsilon_0 c}\left(\frac{\partial y_\nu(t_r)}{\partial x^\mu} - \frac{\partial y_\mu(t_r)}{\partial x^\nu}\right) \tag{9}$$

equation (8) can be rewritten as follows

$$-\frac{\partial \Pi^{\mu\nu}}{\partial x^\nu} = A^\mu \tag{10}$$

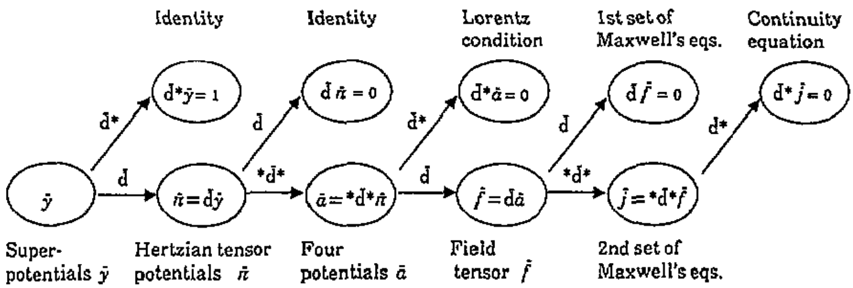

Figure 1. Differential form diagram of electrodynamics.

where equation (5) has been used in equation (10). According to the definition by Nisbet [4] or Laporte and Uhlenbeck [5], the tensor $\Pi_{\mu\nu}$ can be regarded as the Hertzian tensor potentials for the LW potentials.

One can find some similarities between the LW potentials and the superpotentials. We know that the electromagnetic field tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{11}$$

The field tensor satisfies the following Maxwell's equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -\frac{1}{\varepsilon_0 c^2} J^\mu \tag{12}$$

and

$$\frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \tag{13}$$

where $J^\mu = (c\rho, J)$ is the four-dimensional current-density vector. Some similarities between the Hertzian tensor potentials $\Pi_{\mu\nu}$ and the field tensor $F_{\mu\nu}$ can be found by comparing equations (9) and (10) with equations (11) and (12), respectively. Moreover, noting that equation (13) is an identity when tensor $F_{\mu\nu}$ is defined by equation (11), it is readily proved that the Hertzian tensor potentials $\Pi_{\mu\nu}$ satisfy the following equation:

$$\frac{\partial \Pi_{\nu\lambda}}{\partial x^\mu} + \frac{\partial \Pi_{\mu\nu}}{\partial x^\lambda} + \frac{\partial \Pi_{\lambda\mu}}{\partial x^\nu} = 0. \tag{14}$$

Summarizing these correspondences, we have

$$A^\mu \leftrightarrow y^\mu$$
$$F^{\mu\nu} \leftrightarrow \Pi^{\mu\nu}$$
$$J^\mu \leftrightarrow A^\mu.$$

The differential form diagram of electrodynamics—including superpotentials—can be drawn as in figure 1. The notation $*$ denotes the dual operator and $\tilde{d}$ denotes the exterior derivative operator. Figure 1 helps us to understand the relation between the superpotentials $\bar{y}$ and the four potentials $\bar{a}$.

The authors presented a particle trajectory estimation method from far electromagnetic fields using LW superpotentials and confirmed the validity of the estimation method using numerical calculations [6, 7]. The result of the numerical calculation also tells us the validity of the new representation.

3. Consideration on particle dynamics pictures of the Liénard–Wiechert potentials

We have referred to a kind of wave and particle duality of the LW potentials in the introduction of this paper. LW potentials, evidently, are results of the inhomogeneous wave equation. We now have the new representation of the LW potentials. This section discusses particle-dynamical pictures of LW potentials using the new representation.

If the particle motion is periodic, the explicit expression of the LW superpotentials $y^{\mu}(ct, \boldsymbol{x})$ is given by the following Fourier form [2]:

$$y^{\mu}(ct, \boldsymbol{x}) = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{1}{n} \int_0^{2\pi} \exp\left[in\left(\omega_0 t - \sigma - \frac{\omega_0}{c}|\boldsymbol{x} - \boldsymbol{y}(\tau)|\right)\right] \mathrm{d}y^{\mu}(\sigma) \tag{15}$$

where ω_0 is the angular frequency of the periodic motion. For non-periodic motion, equation (15) becomes

$$y^{\mu}(ct, \boldsymbol{x}) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{\omega} \int_{-\infty}^{\infty} \exp \mathrm{i}\left[\omega t - \omega t' - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(t')|\right] \mathrm{d}y^{\mu}(t'). \tag{15'}$$

On the other hand, the new representation of LW potentials (8), gives us the following particular integral:

$$y^{\mu}(ct, \boldsymbol{x}) = \frac{8\pi\varepsilon_0 c}{e} \frac{1}{4\pi} \int_V \mathrm{d}v' A^{\mu}\left(t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}, \boldsymbol{x}'\right) \Big/ |\boldsymbol{x} - \boldsymbol{x}'|. \tag{16}$$

Of course, expressions (15′) and (16) have to be equivalent to each other. To compare equation (15′) with equation (16), we shall transform equation (16) as follows

$$\begin{aligned} y^{\mu}(ct, \boldsymbol{x}) &= \frac{2\varepsilon_0 c}{e} \int_V \mathrm{d}v' A^{\mu}\left(t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}, \boldsymbol{x}'\right) \Big/ |\boldsymbol{x} - \boldsymbol{x}'| \\ &= \frac{2\varepsilon_0 c}{e} \int_V \mathrm{d}v' \int_{-\infty}^{\infty} \mathrm{d}t' \frac{A^{\mu}(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \delta\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\ &= \frac{\varepsilon_0 c}{e\pi} \int_V \mathrm{d}v' \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{A^{\mu}(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right). \end{aligned} \tag{17}$$

Since equations (15′) and (17) are equivalent to each other, we obtain

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \frac{1}{\omega} \int_{-\infty}^{\infty} \exp \mathrm{i}\left[\omega t - \omega t' - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{y}(t')|\right] \frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'} \mathrm{d}t' \\ &\qquad = \frac{\varepsilon_0}{e\pi} \int_{\Omega} \mathrm{d}\Omega' \frac{A^{\mu}(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \end{aligned} \tag{18}$$

where $\mathrm{d}\Omega'(= c\,\mathrm{d}t'\,\mathrm{d}v')$ denotes four-dimensional infinitesimal volume. Equation (18) implies that the line and volume integrals in four-dimensional space are equivalent to each other. However, there are no mathematical theorems that equate line integrals to volume integrals. This aspect reminds us of Dirac's tube in four-dimensional space [8]. That is to say, regarding the line integral in equation (18) as a mean value of the integral on a very small super-surface which encloses the particle orbit (figure 2), we can apply Gauss's theorem to equation (18). For this, we shall transform the line integral (the left-hand side of equation (18)) as follows

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\frac{1}{\omega}\int_{-\infty}^{\infty}\exp\mathrm{i}\left[\omega t-\omega t'-\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{y}(t')|\right]\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'}\,\mathrm{d}t' \\
&\quad=\frac{1}{2\pi \mathrm{i}}\frac{1}{\omega}\int_{-\infty}^{\infty}\frac{1}{S_0}\int_S \mathrm{d}S'\exp\mathrm{i}\left[\omega t-\omega t'-\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{y}(t')|\right]\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'}\,\mathrm{d}t' \\
&\quad\simeq\frac{1}{2\pi c\mathrm{i}}\frac{1}{\omega}\int_V \mathrm{d}V'\frac{1}{S_0}\exp\mathrm{i}\left[\omega t-\omega t'-\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right]\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'}
\end{aligned}\tag{19}$$

where S is a very small two-dimensional surface which encloses the particle and

$$S_0=\int_S \mathrm{d}S'\tag{20}$$

$$\mathrm{d}V'=c\,\mathrm{d}t'\,\mathrm{d}S'\tag{21}$$

(V denotes the super-surface on the tube). The volume integral in equation (18) is regarded as an integral on the outside of the tube. This volume integral is transformed as follows (see appendix A)

$$\begin{aligned}
&\frac{\varepsilon_0}{e\pi}\int_{\Omega}\mathrm{d}\Omega'\frac{A^{\mu}(t',\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\exp\mathrm{i}\omega\left(t-t'-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\
&\quad=\frac{-\varepsilon_0 c}{2\pi e\omega}\int_{\Omega}\mathrm{d}\Omega'\frac{\partial}{\partial x^{\nu\prime}}\left[\left(A^{\mu}\frac{\partial}{\partial x'_{\nu}}\left(\omega t'+\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)-\mathrm{i}\frac{\partial A^{\mu}}{\partial x'_{\nu}}\right)\right. \\
&\quad\quad\left.\times\exp\mathrm{i}\left[\omega t-\omega t'-\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right]\right].
\end{aligned}\tag{22}$$

In this way, Gauss's theorem can be applied to equation (22) as follows

$$\begin{aligned}
&\frac{\varepsilon_0}{e\pi}\int_{\Omega}\mathrm{d}\Omega'\frac{A^{\mu}(t',\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\exp\mathrm{i}\omega\left(t-t'-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \\
&\quad=\frac{\varepsilon_0 c}{2\pi e\omega}\int_V \mathrm{d}V\xi_{\nu}\left[\left(A^{\mu}\frac{\partial}{\partial x'_{\nu}}\left(\omega t'+\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)-\mathrm{i}\frac{\partial A^{\mu}}{\partial x'_{\nu}}\right)\right. \\
&\quad\quad\left.\times\exp\mathrm{i}\left[\omega t-\frac{\omega}{c}t'-\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right]\right]
\end{aligned}\tag{23}$$

where $(-\xi_{\nu})$ denotes a unit normal vector on the tube surface (ξ_{ν} is also normal to the direction of the orbit ($u^{\nu}\xi_{\nu}=0$)). Equations (19) and (23) have to be equivalent to each other, because of equation (18). Comparing the right-hand sides of equations (19) and (23), we derive the following equations:

$$\frac{1}{2\pi c\mathrm{i}\omega}\frac{1}{S_0}\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'}=\frac{\varepsilon_0 c}{2\pi e\omega}\xi_{\nu}\left(A^{\mu}\frac{\partial}{\partial x'_{\nu}}\left(\omega t'+\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)-\mathrm{i}\frac{\partial A^{\mu}}{\partial x'_{\nu}}\right)\tag{24}$$

or

$$A^{\mu}(t',\boldsymbol{x})\xi_{\nu}\frac{\partial}{\partial x'_{\nu}}\left(\omega t'+\frac{\omega}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)=0 \qquad \text{imaginary part}\tag{25}$$

$$\frac{1}{c}\frac{1}{S_0}\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'}=\frac{\varepsilon_0 c}{e}\xi_{\nu}\frac{\partial A^{\mu}(t',\boldsymbol{x})}{\partial x'_{\nu}} \qquad \text{real part.}\tag{26}$$

Here, one should note that equations (25) and (26) are evaluated as mean values on the tube surface, since the left-hand side of equation (18) is transformed into equation (19) based on such an idea. Similarly, all values in equations (25) and (26) have to be evaluated as mean value on the tube surface, for example,

$$\frac{\mathrm{d}y^{\mu}(t')}{\mathrm{d}t'} = \frac{\varepsilon_0 c^2}{e} S_0 \frac{\partial A^{\mu}(t')}{\partial x_{\nu}} \xi_{\nu}(t'). \tag{26'}$$

Changing the variable t' to s ($s = c\tau$; τ is the proper time), we have

$$u^{\mu}(s) = \frac{\varepsilon_0 c}{e} \gamma(s) S_0 \frac{\partial A^{\mu}(s)}{\partial x_{\nu}} \xi_{\nu}(s). \tag{27}$$

One finds that only the anti-symmetric part of the value $\partial A^{\mu}/\partial x^{\nu}$ can contribute to equation (27) (see appendix B). In this way, equation (27) becomes

$$\begin{aligned} u^{\mu}(s) &= -\frac{\varepsilon_0 c}{2e} \gamma(s) S_0 F^{\mu\nu} \xi_{\nu}(s) \\ F^{\mu\nu} &= \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}. \end{aligned} \tag{28}$$

Differentiation of equation (28) by s along the orbit yields

$$\frac{\mathrm{d}u^{\mu}(s)}{\mathrm{d}s} = -\frac{\varepsilon_0 c}{2e} \gamma(s) S_0 F^{\mu\nu} \frac{\mathrm{d}\xi_{\nu}(s)}{\mathrm{d}s} - \frac{\varepsilon_0 c}{2e} S_0 \frac{\mathrm{d}(F^{\mu\nu}\gamma(s))}{\mathrm{d}s} \xi_{\nu}(s). \tag{29}$$

The vector $\xi_{\nu}(s)$ is normal to the orbit direction ($u^{\nu}\xi_{\nu} = 0$). Therefore, $\xi_{\nu}(s)$ can be expanded using the Frenet coordinate as follows

$$\xi_{\nu}(s) = a_1(s)\xi_{\nu}^{(1)}(s) + a_2(s)\xi_{\nu}^{(2)}(s) + a_3(s)\xi_{\nu}^{(3)}(s) \tag{30}$$

where the vectors $\xi_{\nu}^{(1)}$, $\xi_{\nu}^{(2)}$ and $\xi_{\nu}^{(3)}$ are the first, second and third unit-normal vectors and the coefficients $a_1(s)$, $a_2(s)$ and $a_3(s)$ satisfy the following condition:

$$a_1(s)^2 + a_2(s)^2 + a_3(s)^2 = 1. \tag{31}$$

Applying the Frenet–Serret equations to $\mathrm{d}\xi_{\nu}(s)/\mathrm{d}s$, equation (29) becomes

$$\begin{aligned} \frac{\mathrm{d}u^{\mu}}{\mathrm{d}s} = -\frac{\varepsilon_0 c}{2e} S_0 \Bigg[&-a_1\nu_1\gamma(s)F^{\mu\nu}u_{\nu}(s) + \left[a_1 \frac{\mathrm{d}(F^{\mu\nu}\gamma(s))}{\mathrm{d}s} - a_2\nu_2\gamma(s)F^{\mu\nu}\right]\xi_{\nu}^{(1)} \\ &+ \left[a_2\frac{\mathrm{d}(F^{\mu\nu}\gamma(s))}{\mathrm{d}s} - a_1\nu_2\gamma(s)F^{\mu\nu} - a_3\nu_3\gamma(s)F^{\mu\nu}\right]\xi_{\nu}^{(2)} \\ &+ \left[a_3\frac{\mathrm{d}(F^{\mu\nu}\gamma(s))}{\mathrm{d}s} + a_2\nu_3F^{\mu\nu}\gamma(s)\right]\xi^{(3)}\Bigg] \end{aligned} \tag{32}$$

where $\nu_1(s)$, $\nu_2(s)$ and $\nu_3(s)$ are the first, second and third curvatures of the orbit.

Here, if the following condition is satisfied,

$$\frac{\varepsilon_0 c}{2e} S_0 a_1(s)\nu_1(s)\gamma(s) = \frac{e}{mc} \tag{33}$$

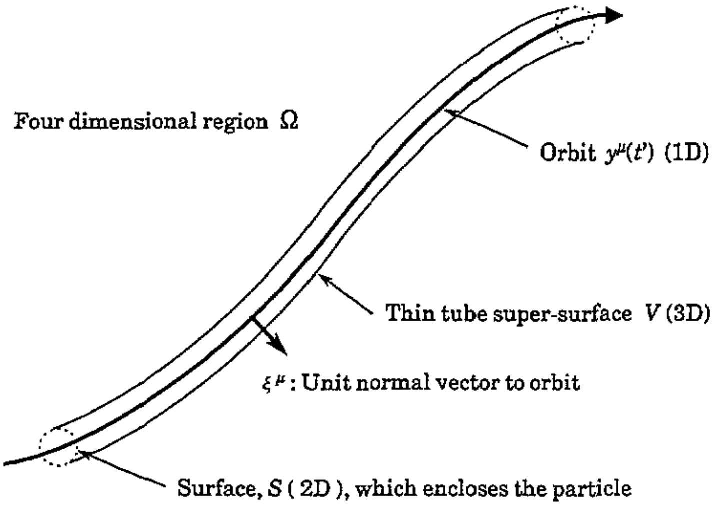

Figure 2. Thin tube model of the particle orbit.

and the sum of the second, third and fourth terms in the right-hand side of equation (32) is very small compared with the first term, equation (32) becomes the well known equation of motion (the Lorentz force equation). However, one cannot proceed to further discussions, because classical electrodynamics in such a small region has not yet been developed. However, it should be remembered that equation (32) was derived from the LW potentials (results of inhomogeneous wave equation), not particle dynamics.

Ringermacher [9] pointed out the Lorentz–Dirac equation for a radiating particle,

$$mc\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = eF^\mu_\nu u^\nu + \frac{e^2}{6\pi\varepsilon_0 c}\left(\frac{\mathrm{d}^2u^\mu}{\mathrm{d}s^2} + u^\mu\frac{\mathrm{d}u_\nu}{\mathrm{d}s}\frac{\mathrm{d}u^\nu}{\mathrm{d}s}\right) \tag{34}$$

can be regarded as an expansion of the vector $eF^\mu_\nu u^\nu$ (which is perpendicular to the vector u^μ) using the basis of the Frenet coordinate

$$eF^\mu_\nu u^\nu = \beta^{(1)}\xi^{(1)\nu} + \beta^{(2)}\xi^{(2)\mu} + \beta^{(3)}\xi^{(3)\mu} \tag{35}$$

where $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$ are the components of the Frenet coordinate and have the following values:

$$\beta^{(1)} = \frac{e^2}{6\pi\varepsilon_0 c}\frac{\mathrm{d}\nu_1}{\mathrm{d}s} + mc\nu_1 \tag{36}$$

$$\beta^{(2)} = \frac{e^2}{6\pi\varepsilon_0 c}\nu_1\nu_2 \tag{37}$$

$$\beta^{(3)} = 0. \tag{38}$$

The third base $\xi^{(3)\mu}$ is not used in the expansion. However, it is natural that all of the bases are used for the expansion, because only condition ($eF^\mu_\nu u^\nu u_\mu = 0$) is imposed on the vector $eF^\mu_\nu u^\nu$. Taking into account that the Lorentz–Dirac equation has some difficulties itself, one can interpret equation (32) as one possible generalization of the Lorentz–Dirac equation. Of course, more information is necessary to find the final coefficient $\beta^{(3)}$, which makes the Lorentz–Dirac equation free from the difficulties.

4. Consideration on the new representation of the Liénard–Wiechert potentials from quantum mechanical point of view

In the last part of this paper, we shall consider the particle-dynamics picture of the LW potentials from a wave mechanics point of view. Indeed, the new representation gives us a relation between electromagnetic fields A^μ and particle coordinates y^μ in the wave equation form. Equation (8) is rewritten as follows, using equation (6)

$$\Box y^\mu = -\frac{8\pi\varepsilon_0 c}{e} A^\nu \frac{\partial}{\partial x^\nu} y^\mu \tag{39}$$

or

$$\frac{\partial}{\partial x^\nu}\left[\frac{\partial}{\partial x_\nu} - \frac{8\pi\varepsilon_0 c}{e} A^\nu\right] y^\mu = 0 \tag{40}$$

where the Lorentz gauge condition,

$$\frac{\partial A^\nu}{\partial x^\nu} = 0 \tag{41}$$

has been invoked. Changing variables from x^μ to $\xi^\mu = x^\mu/\mathrm{i}a$, equation (40) becomes

$$\left[-\hbar^2 \frac{\partial^2}{\partial \xi^\nu \partial \xi_\nu} + 2\mathrm{i}\hbar e A^\nu \frac{\partial}{\partial \xi^\nu}\right] \phi^\mu(\xi^\lambda) = 0 \tag{42}$$

where $a(= e^2/(4\pi\varepsilon_0 c\hbar))$ is the fine structure constant and ϕ^μ is defined by $\phi^\mu(\xi^\lambda) = y^\mu(x^\lambda)$. We shall use the notation $(\partial/\partial x^\mu)$ instead of $(\partial/\partial \xi^\mu)$ from here, for simplicity, i.e.

$$\left[\hbar^2 \Box + 2\mathrm{i}\hbar e A^\nu \frac{\partial}{\partial x^\nu}\right] \phi^\mu = 0. \tag{43}$$

This is reminiscent of the Klein–Gordon equation

$$\left[\hbar^2 \Box + 2\mathrm{i}\hbar e A^\nu \frac{\partial}{\partial x^\nu} - e^2 A^\nu A_\nu + m^2 c^2\right] \psi = 0. \tag{44}$$

In fact, if we replace $\mathrm{i}\hbar\partial/\partial x^\nu$ with the generalized momentum P_ν, equation (43) becomes

$$P_k(P^k - 2eA^k)\psi = 0. \tag{45}$$

Using the relation $P^\nu = p^\nu + eA^\nu$ (where $p^\nu = mu^\nu$ is ordinary momentum), equation (45) is transformed into,

$$(p_\nu + eA_\nu)(p^\nu - eA^\nu)\psi = 0 \tag{46}$$

or

$$(p_\nu p^\nu - e^2 A_\nu A^\nu)\psi = 0 \qquad (m^2c^2 - e^2 A_\nu A^\nu)\psi = 0. \tag{47}$$

Thus, equation (43) implies that the last two terms in equation (44) vanish and, hence, y^μ satisfies the Klein–Gordon equation. Then, the superpotentials y^μ correspond to the wavefunction ψ (or spinor).

Here, it should be noted that one cannot say anything about the relation between the above Klein–Gordon equation and the Klein–Gordon equation in quantum mechanics.

5. Summary

This paper has discussed particle-dynamics pictures revealed by LW potentils and some formulae on the LW superpotentials have been presented.

The particular integral of the new representation has given us an equation of motion which is similar to the Lorentz-force equation of motion. The equation of motion has also given us one possible generalization of the Lorentz–Dirac equation for a radiating particle.

Consideration on wave pictures of the new representation, with respect to the LW superpotentials, has shown the existence of some formulae which are similar to formulae in quantum mechanics.

Appendix A

Using the formula

$$\Box'[ct' + |\boldsymbol{x} - \boldsymbol{x}'|] = \frac{2}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{A1}$$

the right-hand side of equation (22) is transformed as follows,

$$\begin{aligned}
&\frac{\varepsilon_0}{e\pi}\frac{A^\mu(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&\quad = \frac{\varepsilon_0}{2e\pi}A^\mu \Box'[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&\quad = \frac{-\varepsilon_0}{2e\pi}\frac{\partial}{\partial x^{\nu'}}\left[A^\mu \frac{\partial}{\partial x'_\nu}[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right)\right] \\
&\qquad + \frac{\varepsilon_0}{2e\pi}\frac{\partial[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]}{\partial x'_\nu}\left[\frac{\partial A^\mu}{\partial x^{\nu'}} - \mathrm{i}\frac{\omega}{c}A^\mu \frac{\partial[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]}{\partial x^{\nu'}}\right] \\
&\qquad \times \exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right).
\end{aligned} \tag{A2}$$

Noticing that,

$$\frac{\partial[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]}{\partial x^{\nu'}} = \left(1, -\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|}\right) \tag{A3}$$

$$\frac{\partial[ct' + |\boldsymbol{x} - \boldsymbol{x}'|]}{\partial x'_\nu} = \left(1, \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|}\right) \tag{A3$'$}$$

it is found that equation (A2) is rewrittn as follows,

$$\begin{aligned}
&\frac{\varepsilon_0}{e\pi}\frac{A^\mu(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right) \\
&\quad = \frac{\varepsilon_0 c}{2e\pi\omega}\frac{\partial}{\partial x^{\nu'}}\left[\frac{A^\mu}{i}\frac{\partial}{\partial x'_\nu}\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right)\right] \\
&\qquad - \frac{\varepsilon_0 c}{2e\pi\omega}\frac{\partial A^\mu}{\partial x^{\nu'}}\frac{\partial}{\mathrm{i}\partial x'_\nu}\exp \mathrm{i}\omega\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right).
\end{aligned} \tag{A4}$$

Transforming the second term in the right-hand side of equation (A4), we can obtain

$$
\begin{aligned}
\frac{\varepsilon_0}{e\pi}\frac{A^\mu(t',\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\exp \mathrm{i}\omega\left(t-t'-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)& \\
=\frac{-\varepsilon_0 c}{2e\pi\omega}\frac{\partial}{\partial x^{\nu'}}&\left[A^\mu\frac{-1}{\mathrm{i}}\frac{\partial}{\partial x'_\nu}\exp \mathrm{i}\omega\left(t-t'-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)\right. \\
&\left.-\mathrm{i}\frac{\partial A^\mu}{\partial x'_\nu}\exp \mathrm{i}\omega\left(t-t'-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)\right]
\end{aligned}
\qquad (A5)
$$

where the following condition was used in the above transformation:

$$
\Box' A^\mu = 0, \qquad \text{in the integral region } \Omega \text{ of the equation (18).} \qquad (A6)
$$

Appendix B

The value $\partial A^\mu/\partial x_\nu$ can be split into two parts, a symmetrical and an anti-symmetrical parts, as follows,

$$
\begin{aligned}
\frac{\partial A^\mu}{\partial x_\nu}&=\frac{1}{2}\left(\frac{\partial A^\mu}{\partial x_\nu}+\frac{\partial A^\nu}{\partial x_\mu}\right)+\frac{1}{2}\left(\frac{\partial A^\mu}{\partial x_\nu}-\frac{\partial A^\nu}{\partial x_\mu}\right) \\
&=\left(\frac{\partial A^\mu}{\partial x_\nu}\right)_{\mathrm{sym}}+\left(\frac{\partial A^\mu}{\partial x_\nu}\right)_{\mathrm{anti}}.
\end{aligned}
\qquad (B1)
$$

Taking the scalar product between ξ_μ and equation (27)

$$
\begin{aligned}
u^\mu(s)\xi_\mu(s)&=\frac{\varepsilon_0 c}{e}S_0\gamma(s)\frac{\partial A^\mu}{\partial x_\nu}\xi_\nu(s)\xi_\mu(s) \\
0&=\frac{\varepsilon_0 c}{e}S_0\gamma(s)\left(\frac{\partial A^\mu}{\partial x_\nu}\right)_{\mathrm{sym}}\xi_\nu(s)\xi_\mu(s).
\end{aligned}
\qquad (B2)
$$

To satisfy equation (B2) for arbitrary ξ_μ, $\partial A^\mu/\partial x^\nu$ in equation (27) should have only the antisymmetrical component as follows

$$
u^\mu(s)=\frac{\varepsilon_0 c}{2e}S_0\gamma(s)\left(\frac{\partial A^\mu}{\partial x_\nu}-\frac{\partial A^\nu}{\partial x_\mu}\right)\xi_\nu(s). \qquad (B3)
$$

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